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On the explicit analytic solution of Cheng–Chang equation

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Abstract

An analytic technique is applied to solve the free convection over a heated horizontal flat surface embedded in a porous medium. An explicit, totally analytic and uniformly valid solution is given to the so-called Cheng–Chang equation, which agrees well with numerical results.

Keywords: Cheng–Chang equation; Analytic solution; Homotopy analysis method

1. Introduction

Convective heat transfer in porous medium has been intensively studied over the past two decades because of its wide applications including geothermal energy engineering, groundwater pollution transport, nuclear waste disposal, chemical reactors engineering, insulation of buildings and pipes, storage of grain and coal, and so on. The state of art concerning convective heat transfer in porous media has been summarized in the excellent monographs by Nield and Bejan [1] and Ingham and Pop [2].

The similarity solutions for free convective boundary layer flow in a porous medium above a heated horizontal flat surface or below a cooled horizontal flat surface where the wall temperature is power function of the distance from the leading edge, were first considered by Cheng and Chang [3]. The problem has important applications in the assessment of geothermal resources and design of underground energy storage systems. Behavior of the similarity equations were further considered by Chang and Cheng [4], Merkin and Zhang [5], Na and Pop [6]. Chang and Cheng [4] investigated this problem by the method of matched asymptotic expansions in which other effects, such as fluid entrainment were taken into consideration. Merkin and Zhang [5] presented numerical solutions to the similarity equation and revealed that the solutions have a singularity under certain conditions. Na and Pop [6] presented numerical solutions of the similarity equations using the perturbation method in combination with the Shanks transformation. The similarity solutions for free convection in porous media in other configurations have been studied by Cheng and Minkowycz [7], Cheng [8], Ingham and Brown [9], and most recently by Rees and Pop [10], Postelnicu and Pop [11], Banu and Rees [12].

Due to its important applications in many fields described by Cheng [13], a full understanding of the similarity solutions for free convection boundary layer above a horizontal flat surface in a porous medium is meaningful. Although a lots of numerical results have been reported, to our knowledge, no one has reported an explicit, totally analytic, uniformly valid solution for this problem. In this paper, we employ the homotopy analysis method (HAM, see [14–19]) to give such an explicit analytic solution. Unlike perturbation method, the HAM is independent upon small or large parameters, and has been successfully employed to many non-linear problems. All of these applications verify the validity and potential of the HAM as a kind of powerful analytic tool for non-linear problems [14–19].

In this paper, we apply the HAM to give, for the first time (to the best of our knowledge), explicit analytic solutions of the similarity equations appeared in free

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convection above a horizontal plane in porous medium. The validity of the analytic solution is verified by numerical results.

2. Mathematical formulation

Assuming that the flow is governed by Darcy's law and that the boundary layer approximation holds, the dimensionless equations governing the free convective boundary-layer flow above a heated impermeable horizontal surface (or below a cooled impermeable horizontal surface) are [3]:

$$\frac{\partial^2\psi}{\partial y^2} = -\frac{\partial\theta}{\partial x}, \tag{1}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}, \tag{2}$$

where (x, y) are the dimensionless Cartesian coordinates along and normal to the flat surface, respectively, θ is the dimensionless temperature and ψ is the dimensionless stream function. The minus sign in Eq. (1) corresponds to a flat surface heated upward. Assuming a power-law variation of temperature on the impermeable surface, the boundary condition can be given by

$$\psi = 0, \quad \theta = x^\alpha, \quad \text{on } y = 0, \tag{3}$$

$$\frac{\partial\psi}{\partial y} = 0, \quad \theta = 0, \quad \text{as } y \to \infty. \tag{4}$$

Under the transformation

$$\theta = x^\alpha g(\eta), \quad \psi = (1+\alpha)^{-1/3} x^{(\alpha+1)/3} f(\eta),$$
$$\eta = (1+\alpha)^{2/3} x^{(\alpha-2)/3} y, \tag{5}$$

we have

$$f''(\eta) + \left(\beta - \tfrac{2}{3}\right)\eta g'(\eta) + \beta g(\eta) = 0, \tag{6}$$

$$g''(\eta) + \tfrac{1}{3} f(\eta) g'(\eta) - \beta f'(\eta) g(\eta) = 0, \tag{7}$$

subject to the boundary conditions

$$f(0) = 0, \quad g(0) = 1, \quad f'(\infty) = g(\infty) = 0, \tag{8}$$

where

$$\beta = \frac{\alpha}{1+\alpha}. \tag{9}$$

Now, we solve Eqs. (6)–(8) by means of the HAM. Due to the boundary condition, Eq. (8), $f(\eta)$ and $g(\eta)$ can be expressed by a set of base functions

$$\{\eta^m \exp(-n\eta) \mid n \geqslant 1, m \geqslant 0\} \tag{10}$$

in the forms

$$f(\eta) = A_{0,0} + \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} A_{n,m}\eta^m \exp(-n\eta), \tag{11}$$

$$g(\eta) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} B_{n,m}\eta^m \exp(-n\eta), \tag{12}$$

respectively, where $A_{n,m}$ and $B_{n,m}$ are coefficients. It is straightforward to choose

$$f_0(\eta) = 1 - \exp(-\eta) - \eta\exp(-\eta), \tag{13}$$

$$g_0(\eta) = \exp(-\eta) + \eta\exp(-\eta) \tag{14}$$

as the initial approximations of $f(\eta)$ and $g(\eta)$. We choose

$$L_f = \frac{\partial^2}{\partial\eta^2} - \frac{\partial}{\partial\eta}, \tag{15}$$

$$L_g = \frac{\partial^2}{\partial\eta^2} - 1 \tag{16}$$

as our auxiliary linear operators, which have the properties:

$$L_f[C_1 + C_2\exp(\eta)] = 0, \tag{17}$$

$$L_g[C_3\exp(-\eta) + C_4\exp(\eta)] = 0, \tag{18}$$

where C_1, C_2, C_3 and C_4 are all constants.

We construct the so-called zeroth-order deformation equations

$$(1-p)L_f[F(\eta;p) - f_0(\eta)] = p\hbar_f N_f[F(\eta;p), G(\eta;p)], \tag{19}$$

$$(1-p)L_g[G(\eta;p) - g_0(\eta)] = p\hbar_g N_g[F(\eta;p), G(\eta;p)], \tag{20}$$

subject to the boundary conditions

$$F(0;p) = 0, \quad G(0;p) = 1,$$
$$\left.\frac{\partial F(\eta;p)}{\partial\eta}\right|_{\eta=\infty} = G(\infty;p) = 0, \tag{21}$$

under the definitions

$$N_f[F(\eta;p), G(\eta;p)] = \frac{\partial^2 F(\eta;p)}{\partial\eta^2} + \left(\beta - \frac{2}{3}\right)\eta\frac{\partial G(\eta;p)}{\partial\eta} + \beta G(\eta;p), \tag{22}$$

$$N_g[F(\eta;p), G(\eta;p)] = \frac{\partial^2 G(\eta;p)}{\partial\eta^2} + \frac{1}{3}F(\eta;p)\frac{\partial G(\eta;p)}{\partial\eta} - \beta\frac{\partial F(\eta;p)}{\partial\eta}G(\eta;p), \tag{23}$$

where p is an embedding parameter, $\hbar_f$ and $\hbar_g$ are the auxiliary non-zero parameters. Obviously,

$$F(\eta;0) = f_0(\eta), \quad G(\eta;0) = g_0(\eta), \tag{24}$$

and

$$F(\eta;1) = f(\eta), \quad G(\eta;1) = g(\eta), \tag{25}$$

when $p = 0$ and $p = 1$, respectively.

We expand $F(\eta;p)$ and $G(\eta;p)$ in Taylor's power series at $p = 0$. If the series are convergent at $p = 1$, due to Eqs. (24) and (25) we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{+\infty} f_m(\eta), \tag{26}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{+\infty} g_m(\eta), \tag{27}$$

where

$$f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m F(\eta;p)}{\partial p^m} \right|_{p=0}, \tag{28}$$

$$g_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m G(\eta;p)}{\partial p^m} \right|_{p=0}, \tag{29}$$

being governed by the corresponding mth-order deformation equations [16,17] as

$$L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f R_m(\eta), \tag{30}$$

$$L_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar_g S_m(\eta), \tag{31}$$

subjected to the boundary conditions

$$f_m(0) = 0, \quad g_m(0) = 0, \quad f'_m(\infty) = g_m(\infty) = 0 \tag{32}$$

with

$$R_m(\eta) = f'''_{m-1}(\eta) + \left(\beta - \tfrac{2}{3}\right)\eta g'_{m-1}(\eta) + \beta g_{m-1}(\eta), \tag{33}$$

$$S_m(\eta) = g''_{m-1} + \frac{1}{3}\sum_{n=0}^{m-1} f_n(\eta) g'_{m-1-n}(\eta) - \beta \sum_{n=0}^{m-1} f'_n(\eta) g_{m-1-n}(\eta) \tag{34}$$

and

$$\chi_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases} \tag{35}$$

It is found that $f_m(\eta)$ and $g_m(\eta)$ governed by Eqs. (30)–(32) can be expressed by

$$f_m(\eta) = a^0_{m,0} + \sum_{k=1}^{m+1}\sum_{i=0}^{m+1} a^i_{m,k}\eta^i \exp(-k\eta), \tag{36}$$

$$g_m(\eta) = \sum_{k=1}^{m+1}\sum_{i=0}^{m+1} b^i_{m,k}\eta^i \exp(-k\eta) \tag{37}$$

for $m \geqslant 1$, where $a^i_{m,k}$ and $b^i_{m,k}$ are coefficients. Substituting above expressions into Eqs. (30)–(32), we have the recursive formulae

$$a^i_{m,k} = \hbar_f \sum_{j=i}^{m+1} \mu^i_{k,j}(d^j_{m-1,k} + \beta\lambda^j_{m-1,k} b^j_{m-1,k}) + \hbar_f \sum_{j=\max\{0,i-1\}}^{m} \left(\beta - \frac{2}{3}\right)\mu^i_{k,j+1} e^j_{m-1,k} + \chi_m \lambda^i_{m-1,k} a^i_{m-1,k} \tag{38}$$

for $1 \leqslant k \leqslant m+1$ and $0 \leqslant i \leqslant m+1$,

$$a^0_{m,0} = -\sum_{k=1}^{m+1} a^0_{m,k}, \tag{39}$$

and

$$b^i_{m,k} = \chi_m \lambda^i_{m-1,k} b^i_{m-1,k} + \hbar_g \sigma^i_{m,k} \tag{40}$$

for $k = 1$ and $1 \leqslant i \leqslant m+1$, or $2 \leqslant k \leqslant m+1$ and $0 \leqslant i \leqslant m+1$, and

$$b^0_{m,1} = (-1)\sum_{j=2}^{m+1}(\chi_m \lambda^0_{m-1,j} b^0_{m-1,j} + \hbar_g \sigma^0_{m,j}), \tag{41}$$

Here

$$\mu^i_{n,q} = \sum_{j=i}^{q}\left(\frac{q!}{i!}\right)\frac{1}{(n+1)^{q-j+1}}\frac{1}{n^{j-i+1}}, \quad 0 \leqslant i \leqslant q, \tag{42}$$

$$\lambda^i_{m,k} = \begin{cases} 1, & 0 \leqslant i \leqslant m+1 \text{ and } 1 \leqslant k \leqslant m+1, \\ 0, & \text{otherelse}, \end{cases} \tag{43}$$

and

$$d^i_{m,k} = (i+2)(i+1)\lambda^{i+2}_{m,k} a^{i+2}_{m,k} - 2k(i+1)\lambda^{i+1}_{m,k} a^{i+1}_{m,k} + k^2 \lambda^i_{m,k} a^i_{m,k}, \tag{44}$$

$$e^i_{m,k} = (i+1)\lambda^{i+1}_{m,k} b^{i+1}_{m,k} - k\lambda^i_{m,k} b^i_{m,k} \tag{45}$$

for $0 \leqslant i \leqslant m+1$ and $1 \leqslant k \leqslant m+1$,

$$\sigma^j_{m,k} = \sum_{q=j}^{m+1}(\tau^q_{m-1,k} + \Gamma^q_{m,k} + \Phi^q_{m,k} + \Delta^q_{m,k})\delta^j_{k,q} \tag{46}$$

for $2 \leqslant k \leqslant m+1$, $0 \leqslant j \leqslant m+1$,

$$\sigma^j_{m,1} = \sum_{i=\max\{0,j-1\}}^{m}(\tau^i_{m-1,1} + \Phi^i_{m,1})\delta^j_{1,i} \tag{47}$$

for $0 \leqslant j \leqslant m+1$, under the definitions

$$\delta^j_{n,q} = \begin{cases} -\frac{q!}{j!}\frac{1}{2^{q-j+2}}, & 0 \leqslant j \leqslant q \text{ and } n = 1, \\ -\frac{1}{2(q+1)}, & j = q+1 \text{ and } n = 1, \\ -\frac{q!}{j!}\left(\frac{1}{2(n+1)^{q-j+1}} - \frac{1}{2(n-1)^{q-j+1}}\right), & 0 \leqslant j \leqslant q \text{ and } n > 1, \\ 0, & j = q+1 \text{ and } n > 1, \end{cases} \tag{48}$$

and

$$\tau^i_{m,k} = (i+2)(i+1)\lambda^{i+2}_{m,k} b^{i+2}_{m,k} - 2k(i+1)\lambda^{i+1}_{m,k} b^{i+1}_{m,k} + k^2 \lambda^i_{m,k} b^i_{m,k} \tag{49}$$

for $0 \leqslant i \leqslant m+1$ and $1 \leqslant k \leqslant m+1$,

$$\Gamma^q_{m,n} = \frac{1}{3}\sum_{k=0}^{m-1}\sum_{j=\max\{1,n-m+k\}}^{\min\{k+1,n-1\}} \times \sum_{i=\max\{0,q-m+k\}}^{\min\{k+1,q\}} a^{q-i}_{m-k-1,n-j}\lambda^{q-i}_{m-1-k,n-j}e^i_{k,j} \tag{50}$$

for $2 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant m+1$,

$$\Phi^i_{m,j} = \frac{1}{3}\sum_{k=\max\{j-1,i-1\}}^{m-1} a^0_{m-k-1,0}e^i_{k,j} \tag{51}$$

for $1 \leqslant j \leqslant m+1$, $0 \leqslant i \leqslant m+1$,

$$\Delta^q_{m,n} = -\beta\sum_{k=0}^{m-1}\sum_{j=\max\{1,n-k-1\}}^{\min\{m-k,n-1\}} \times \sum_{i=\max\{0,q-k-1\}}^{\min\{m-k,q\}} c^{q-i}_{k,n-j}\lambda^i_{m-1-k,j}b^i_{m-1-k,j} \tag{52}$$

for $2 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant m+1$, and

$$c^i_{m,k} = (i+1)\lambda^{i+1}_{m,k}a^{i+1}_{m,k} - k\lambda^i_{m,k}a^i_{m,k} \tag{53}$$

for $0 \leqslant i \leqslant m+1$ and $1 \leqslant k \leqslant m+1$.

Using above recursive formulae, we can calculate all coefficients $a^k_{m,n}$ and $b^k_{m,n}$ by using only the first five: $a^0_{0,0} = 1$, $a^0_{0,1} = -1$, $a^1_{0,1} = -1$, $b^0_{0,1} = 1$, $b^1_{0,1} = 1$, given by the initial approximations (13) and (14). The corresponding Mth-order approximations of (26) and (27) are

$$f(\eta) \approx f_0(\eta) + \sum_{m=1}^{M} f_m(\eta) = \sum_{m=0}^{M} a^0_{m,0} + \sum_{m=0}^{M}\sum_{k=1}^{m+1}\sum_{i=0}^{m+1} a^i_{m,k}\eta^i\exp(-k\eta), \tag{54}$$

$$g(\eta) \approx g_0(\eta) + \sum_{m=1}^{M} g_m(\eta) = \sum_{m=0}^{M}\sum_{k=1}^{m+1}\sum_{i=0}^{m+1} b^i_{m,k}\eta^i\exp(-k\eta). \tag{55}$$

When $M \to +\infty$, we have an *explicit*, *totally analytic* solution of Eqs. (6)–(8).

3. Results analysis

In this section, we verify our analytic solutions by the numerical results given by the finite difference method

Table 1
Parameters used in our analytic approach

α	β	$\hbar_f$	$\hbar_g$	Order M
0	0	−1	−0.2	90
1	0.5	−1	−1	40
3	0.75	−1	−1	40
10	0.909	−1	−1	40
$+\infty$	1	−1	−1	40

Table 2
Analytical results of $f(\eta)$ at different order of approximation compared with numeric results in case of $\alpha = 1$

η	5th order	10th order	15th order	20th order	25th order	30th order	35th order	40th order	Numeric results
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.8294	0.7692	0.7799	0.7777	0.7784	0.7781	0.7782	0.7782	0.7782
2	1.3237	1.2249	1.2447	1.2407	1.2420	1.2415	1.2417	1.2416	1.2416
3	1.6134	1.4921	1.5187	1.5135	1.5152	1.5147	1.5148	1.5148	1.5148
4	1.7814	1.6471	1.6791	1.6732	1.6751	1.6745	1.6747	1.6746	1.6746
5	1.8777	1.7362	1.7723	1.7660	1.7681	1.7674	1.7676	1.7675	1.7676
6	1.9325	1.7868	1.8261	1.8196	1.8218	1.8212	1.8213	1.8213	1.8213
7	1.9632	1.8153	1.8570	1.8503	1.8527	1.8521	1.8522	1.8522	1.8522
8	1.9802	1.8310	1.8747	1.8679	1.8704	1.8697	1.8699	1.8699	1.8699
9	1.9895	1.8397	1.8847	1.8779	1.8805	1.8798	1.8800	1.8800	1.8799
10	1.9945	1.8444	1.8903	1.8835	1.8862	1.8855	1.8857	1.8857	1.8857
11	1.9971	1.8470	1.8935	1.8867	1.8894	1.8887	1.8889	1.8889	1.8889
12	1.9985	1.8483	1.8953	1.8884	1.8912	1.8905	1.8907	1.8907	1.8907
13	1.9992	1.8490	1.8963	1.8894	1.8922	1.8915	1.8917	1.8917	1.8917
14	1.9996	1.8494	1.8968	1.8900	1.8927	1.8921	1.8923	1.8922	1.8922
15	1.9997	1.8496	1.8971	1.8902	1.8931	1.8924	1.8926	1.8925	1.8925
16	1.9998	1.8497	1.8973	1.8904	1.8932	1.8926	1.8927	1.8927	1.8927
17	1.9999	1.8498	1.8973	1.8905	1.8933	1.8926	1.8928	1.8928	1.8928
18	1.9999	1.8498	1.8974	1.8905	1.8933	1.8927	1.8928	1.8928	1.8928
19	1.9999	1.8498	1.8974	1.8905	1.8934	1.8927	1.8929	1.8928	1.8928
20	1.9999	1.8498	1.8974	1.8905	1.8934	1.8927	1.8929	1.8928	1.8928

Table 3
Analytical results of $g(\eta)$ at different order of approximation compared with numeric results in case of $\alpha = 1$

η	5th order	10th order	15th order	20th order	25th order	30th order	35th order	40th order	Numeric results
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.4834	0.5097	0.5035	0.5050	0.5044	0.5046	0.5045	0.5046	0.5046
2	0.2366	0.2653	0.2573	0.2593	0.2585	0.2588	0.2587	0.2587	0.2587
3	0.1175	0.1403	0.1330	0.1350	0.1341	0.1344	0.1343	0.1344	0.1343
4	0.0593	0.0749	0.0693	0.0709	0.0702	0.0705	0.0704	0.0704	0.0704
5	0.0305	0.0401	0.0363	0.0375	0.0370	0.0372	0.0371	0.0371	0.0371
6	0.0160	0.0214	0.0191	0.0199	0.0195	0.0197	0.0196	0.0196	0.0196
7	0.0085	0.0114	0.0100	0.0106	0.0103	0.0104	0.0104	0.0104	0.0104
8	0.0046	0.0060	0.0053	0.0056	0.0055	0.0055	0.0055	0.0055	0.0055
9	0.0025	0.0031	0.0028	0.0030	0.0029	0.0029	0.0029	0.0029	0.0029
10	0.0013	0.0016	0.0015	0.0016	0.0015	0.0016	0.0016	0.0016	0.0016

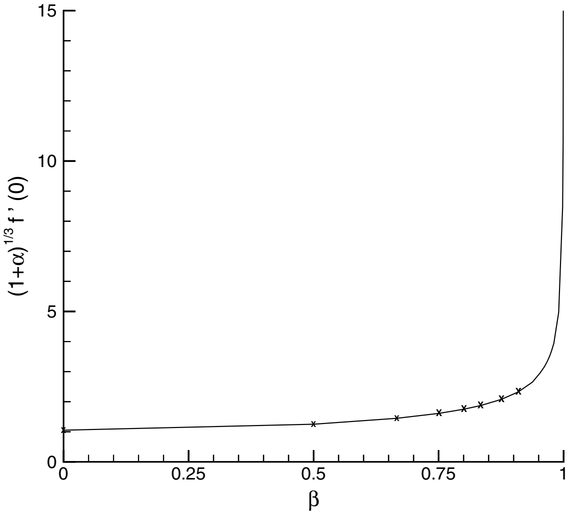

Fig. 1. Analytic results of $(1+\alpha)^{\frac{1}{3}}f'(0)$ compared with numerical ones given by Na and Pop [6]. Symbol: numerical results; solid line: homotopy analysis results.

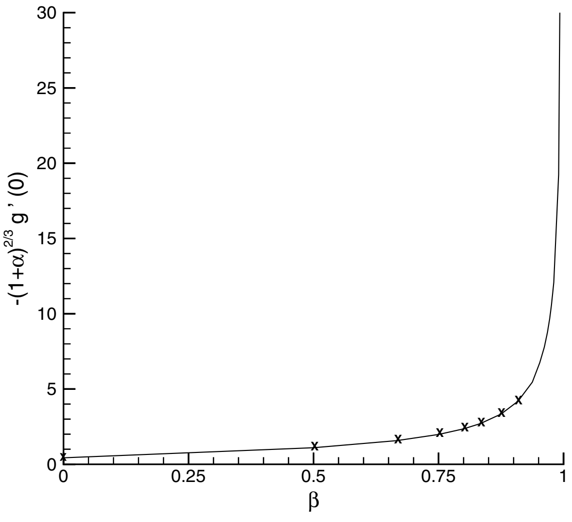

Fig. 2. As Fig. 1, but for $-(1+\alpha)^{\frac{2}{3}}g'(0)$.

[3,6]. Parameters used in our analytic approach are listed in Table 1. It is found that our analytic approximations agree well with the numerical ones, as shown in Tables 2 and 3, and in Figs. 1 and 2.

As shown in Figs. 3 and 4, our analytic results agree well with numerical ones in a wide range of parameter β. Note that solution for $\beta = 1$, corresponding to $\alpha \to +\infty$, is also given. Due to the singularity at $\alpha = -2/5$ reported by Merkin and Zhang [5], higher order of approximation is needed when $\alpha = 0$, as shown in Table 1.

The convergence of the HAM series is controlled by the two important parameter $\hbar_f$ and $\hbar_g$. The parameter $\hbar_g$ can be set to -1 for $\beta \geqslant 0.5$, and dropped for $\beta < 0.5$, as shown in Table 1.

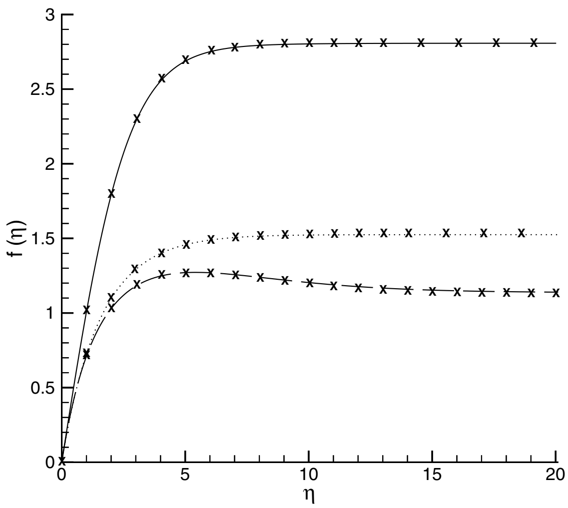

Fig. 3. Homotopy analytic results for $f(\eta)$ compared with numerical ones. Symbol: numerical results; solid line: homotopy analysis result when $\beta = 0$; dotted line: homotopy analysis result when $\beta = 0.75$; long dash line: homotopy analysis result when $\beta = 1$.

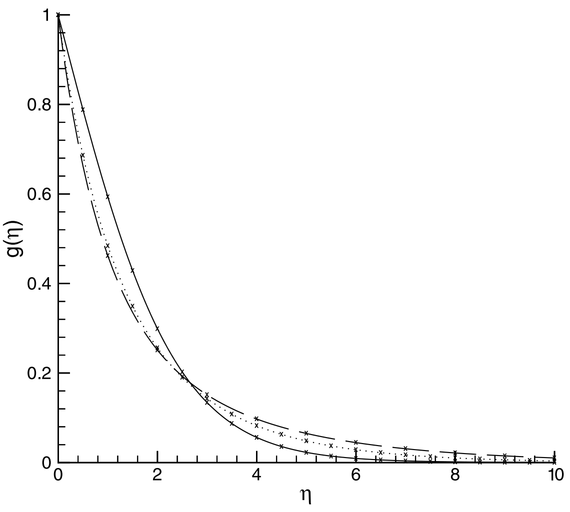

Fig. 4. As Fig. 3 but for $g(\eta)$.

4. Conclusions

We apply the HAM in this paper to obtain an explicit, totally analytic, uniformly valid solution to the so-called Cheng–Chang equation appeared in free convection in porous medium above a horizontal flat surface.

The validity of our analytic solution is verified by numeric results. To the best of our knowledge, it is the first time that such an explicit, uniformly valid, totally analytic solution to Cheng–Chang equation is given. This explicit analytic solution might find wide applications in geothermal energy industries, groundwater pollution transport, nuclear waste disposal, chemical reactors engineering, insulation of buildings and pipes, and storage of grain and coal, and so on.

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